Journal of Pure and Applied Mathematics: Advances and Applications
Volume 2, Number 2, 2009, Pages 209-226

ON THREE-POINT BOUNDARY VALUE PROBLEMS WITH NONLINEAR SOURCE TERMS CONTAINING FIRST ORDER DERIVATIVES

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Abstract

We establish existence results of the following three-point boundary value problems:

$$\begin{cases} u''(t) + f(t, u(t), u'(t)) = 0, \ t \in (0, 1), \\ (BC)\, u(0) = 0, \ u(1) = \delta u(\eta), \end{cases}$$

where $0 < \eta < 1$, and $\delta > 0$ with $\delta\eta < 1$ or with $\delta \leq 1$. The approach applied in this paper is upper and lower solution method associated with basic degree theory or Schauder's fixed point theorem. Under the Nagumo's condition posed on the source term, we deal with diverse solutions for this boundary value problem with the function *f* which is continuous, Carathéodory on its domain, respectively.

1. Introduction

In this paper, we consider three-point boundary value problem

2000 Mathematics Subject Classification: Primary 34B15.

Keywords and phrases: upper (lower) solution, (strict) $W^{2,1}$-upper (lower) solution, three-point boundary value problems, Schauder's fixed point theorem, degree theory, Nagumo's condition.

Received November 2, 2009

$$u''(t) + f(t, u(t), u'(t)) = 0, \ t \in (0, 1), \tag{1}$$

$$u(0) = 0, \ u(1) = \delta u(\eta), \tag{2}$$

where $0 < \eta < 1$, and $\delta > 0$ with $\delta\eta < 1$ or with $\delta \leq 1$.

There are plenty of works having appeared on three-point boundary value problems, when f is independent of u', see, for example, [4, 6, 7]. Note that in this article, we are particularly interested in the source term f, which depends on u'. Recently, many authors pay attention to such problems and get existence results of multiple solutions, see [3, 5].

We will discuss the existence of solutions of some general types on three-point boundary value problems by using upper and lower solution method associated with basic degree theory or Schauder's fixed point theorem. This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, for the case f is continuous, we use the classical space $C^1[0, 1]$ of continuous functions on $[0, 1]$ with C^1-norm, and apply Schauder's fixed point theorem on this function space to show the existence of classical solutions in Theorem 1. In Section 4, focusing on the source term f, which is a Carathéodory function, we consider the Sobolev space $W^{2,1}(0, 1)$ defined by

$$W^{2,1}(0, 1) := \{u \in C^1[0, 1] \,|\, u'' \in L^1(0, 1)\}.$$

Two kinds of upper and lower solutions are introduced there, and by applying Schauder's fixed point theorem and degree theory, we get the existence of $W^{2,1}$-solution in Theorems 2 and 3, respectively. Finally, some examples are given in the last section.

2. Preliminaries

Define $G : [0, 1] \times [0, 1] \to (-\infty, \infty)$ by

$$G(t, s) := \frac{1}{1 - \delta\eta} t(1 - s) - U(t, s) - \frac{\delta}{1 - \delta\eta} V(t, s), \ 0 \leq t, \ s \leq 1, \tag{3}$$

where δ and η are given as (2), and

$$U(t, s) = \begin{cases} t - s, & s \le t, \\ 0, & t \le s, \end{cases} \quad V(t, s) = \begin{cases} t(\eta - s), & s \le \eta, \\ 0, & \eta \le s. \end{cases}$$

By direct computations, we get the following lemma.

Lemma 1. (i) *The function* $G : [0, 1] \times [0, 1] \to (-\infty, \infty)$ *defined by* (3), *is the Green function corresponding for the problem*

$$\begin{cases} u''(t) = 0, \\ u(0) = 0, \; u(1) = \delta u(\eta). \end{cases}$$

(ii) *The function* $G : [0, 1] \times [0, 1] \to (-\infty, \infty)$ *defined by* (3), *is continuous.*

3. Existence of Classical Solutions

In this section, we deal with the classical case, that is, our source term f is continuous, and assume $0 < \delta\eta < 1$ on our boundary condition (2). The notion of upper and lower solutions are given as follows:

Definition 1. A function $\alpha \in C^2(0, 1) \cap C[0, 1]$ is a lower solution of problem (1), (2), if it satisfies:

(i) $\alpha(0) \le 0, \; \alpha(1) \le \delta\alpha(\eta),$ and

(ii) for all $t \in (0, 1), \; \alpha''(t) + f(t, \alpha(t), \alpha'(t)) \ge 0.$

Definition 2. A function $\beta \in C^2(0, 1) \cap C[0, 1]$ is an upper solution of problem (1), (2), if it satisfies:

(i) $\beta(0) \ge 0, \; \beta(1) \ge \delta\beta(\eta),$ and

(ii) for all $t \in (0, 1), \; \beta''(t) + f(t, \beta(t), \beta'(t)) \le 0.$

Definition 3. Let α be a lower solution and β be an upper solution for problem (1), (2), satisfying $\alpha \le \beta$ on $[0, 1]$. We say that a continuous function f satisfies Nagumo's condition with respective to α and β, if there exists a function $h \in C([0, \infty); (0, +\infty))$ such that

$$|f(t, u, v)| \le h(|v|),$$

for all $(t, u, v) \in [0, 1] \times [\alpha(t), \beta(t)] \times \mathbb{R}$, and

$$\int_0^\infty \frac{s}{h(s)} ds = \infty.$$

The proofs for the existence theorems for solutions of boundary value problems depend on finding a priori bounds for the solution and its derivative. Hence, we need the following well-known result.

Lemma 2 [2]. *Assume that f is a continuous function satisfying Nagumo's condition on $[0, 1]$ with respect to α and β. Then for any solution $u(t) \in C^2[0, 1]$ of (1) with $\alpha(t) \le u(t) \le \beta(t)$ on $[0, 1]$, there exists an $L > 0$ depending only on α, β, h such that*

$$|u'(t)| \le L \text{ on } [0, 1].$$

It follows from Lemma 2 that we choose a number N large enough and two functions $\tilde{f}, \tilde{\tilde{f}} : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via

$$\tilde{f}(t, u, v) = \begin{cases} f(t, u, -N), & \text{if } v < -N, \\ f(t, u, v), & \text{if } -N \le v \le N, \\ f(t, u, N), & \text{if } v > N, \end{cases}$$

and

$$\tilde{\tilde{f}}(t, u, v) = \begin{cases} \tilde{f}(t, \alpha(t), v) + \frac{\alpha(t) - u}{1 + |\alpha(t)| + |u|}, & \text{if } u < \alpha(t), \\ \tilde{f}(t, u, v), & \text{if } \alpha(t) \le u \le \beta(t), \\ \tilde{f}(t, \beta(t), v) + \frac{\beta(t) - u}{1 + |\beta(t)| + |u|}, & \text{if } u > \beta(t). \end{cases}$$

Note that $\tilde{\tilde{f}}$ is a continuous function on $[0, 1] \times \mathbb{R} \times \mathbb{R}$, satisfying, for some $M > 0$,

$$|\tilde{\tilde{f}}(t, u, v)| \le M, \text{ for } (t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R}. \tag{4}$$

Before proving our main results, we first focus on such a modified problem given as follows:

$$u''(t) + \tilde{\tilde{f}}(t, u, u') = 0, \; t \in (0, 1), \tag{5}$$

with boundary condition (2).

Proposition 1. *Let* $\alpha(t)$ *and* $\beta(t)$ *be the respective lower and upper solution of problem* (1), (2) *with* $\alpha(t) \leq \beta(t)$ *on* $[0, 1]$, *and let* f *be continuous on E and satisfy Nagumo's condition with respect to* α *and* β *on* $[0, 1]$, *where*

$$E := \{(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \mid \alpha(t) \leq u \leq \beta(t)\}.$$

Moreover, we assume that on domain E, for each fixed (t, u), $f(t, u, v)$ *is nondecreasing in v. If* $u \in C^2(0, 1) \cap C[0, 1]$ *is a solution of the modified problem* (5), (2), *then* $\alpha(t) \leq u(t) \leq \beta(t)$, *for any* $t \in [0, 1]$.

Proof. Let us assume on the contrary that, for some $t_0 \in [0, 1]$,

$$\min_{t \in [0,1]} (u(t) - \alpha(t)) = u(t_0) - \alpha(t_0) < 0.$$

Case (i). If $t_0 \in (0, 1)$, by the definition of lower solution $\alpha(t)$ and $\tilde{\tilde{f}}(t, u)$, we obtain

$$\begin{aligned} 0 &\leq u''(t_0) - \alpha''(t_0) \\ &\leq -\tilde{\tilde{f}}(t_0, u(t_0), u'(t_0)) + f(t_0, \alpha(t_0), \alpha'(t_0)) \\ &= -f(t_0, \alpha(t_0), \alpha'(t_0)) - \frac{\alpha(t_0) - u(t_0)}{1 + |\alpha(t_0)| + |u(t_0)|} + f(t_0, \alpha(t_0), \alpha'(t_0)) \\ &= \frac{\alpha(t_0) - u(t_0)}{1 + |\alpha(t_0)| + |u(t_0)|} < 0. \end{aligned}$$

It leads to a contradiction.

Case (ii). If $t_0 = 0$, by the definition of lower solution, $\alpha(0) \leq 0$, we then have

$$0 = u(0) \leq u(0) - \alpha(0) < 0,$$

and get a contradiction.

Case (iii). If $t_0 = 1$, let $w(t) := u(t) - \alpha(t)$. Note that $w(1) < 0$ and $w(0) \geq 0$. Hence, there exists $\sigma \in [0, 1)$ such that $w(\sigma) = 0$ and $w(t) < 0$, for all $t \in (\sigma, 1]$. We separate it into two subcases. If $\sigma \in (\eta, 1)$, it follows from $w(\eta) < 0$, that there exists $t_1 \in (0, \sigma)$ such that $w(t_1) = \min\{w(t) | t \in [0, \sigma]\}$. Hence, we have $w'(t_1) = 0$ and $w''(t_1) \geq 0$. Therefore,

$$\begin{aligned} 0 &\leq w''(t_1) = u''(t_1) - \alpha''(t_1) \\ &= -f(t_1, \alpha(t_1), \alpha'(t_1)) - \frac{\alpha(t_1) - u(t_1)}{1 + |\alpha(t_1)| + |u(t_1)|} - \alpha''(t_1) \\ &< 0, \end{aligned}$$

which is a contradiction. If $\sigma \in (0, \eta)$, we split the rest into two parts:

(1^0) $w'(t) \leq 0$, for all $t \in [\sigma, 1]$. In this case, for all $t \in (\sigma, 1]$, it follows from $w(t) < 0$, and the nondecreasing property of $f(t, u, v)$ on E, that

$$w''(t) \leq -f(t, \alpha(t), \alpha'(t)) - \frac{\alpha(t) - u(t)}{1 + |\alpha(t)| + |u(t)|} - \alpha''(t) < 0.$$

Hence, $w(t)$ is concave on $[\sigma, 1]$, which implies

$$\frac{w(\eta)}{\eta - \sigma} > \frac{w(1)}{1 - \sigma}.$$

However, $w(1) \geq \delta w(\eta) > \frac{1}{\eta} w(\eta)$, which is impossible.

(2^0) there exists $t_2 \in (\sigma, 1)$ such that $w(t_2) < 0, w'(t_2) = 0$, and $w''(t_2) \geq 0$. In this case, we have

$$0 \leq w''(t_2) = -f(t_2, \alpha(t_2), \alpha'(t_2)) - \frac{\alpha(t_2) - u(t_2)}{1 + |\alpha(t_2)| + |u(t_2)|} - \alpha''(t_2) < 0,$$

which is a contradiction.

Similarly, we can show that $u(t) \leq \beta(t)$, for any $t \in [0, 1]$, hence, we complete this proof. $\square$

Theorem 1. *Let* $\alpha(t)$ *and* $\beta(t)$ *be the respective lower and upper solution of problem* (1), (2) *with* $\alpha(t) \leq \beta(t)$ *on* $[0, 1]$, *and let* f *be* *continuous on E and satisfy Nagumo's condition with respect to* α *and* β *on* $[0, 1]$, *where*

$$E := \{(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \mid \alpha(t) \leq u \leq \beta(t)\}.$$

Moreover, we assume that on domain E, for each fixed (t, u), $f(t, u, v)$ *is nondecreasing in v. Then, the problem* (1), (2) *has at least one solution* $u \in C^2(0, 1) \cap C[0, 1]$ *such that, for all* $t \in [0, 1]$,

$$\alpha(t) \leq u(t) \leq \beta(t).$$

Proof. Consider the modified problem (5), (2). Define $T : C^1[0, 1] \to C^1[0, 1]$ by

$$(Tu)(t) := \int_0^1 G(t, s)\tilde{\tilde{f}}(s, u(s), u'(s))ds, \tag{6}$$

for $u \in C^1[0, 1]$, where $G(t, s)$ is defined as (3).

Let

$$D := \{u \in C^1[0, 1] \mid \|u\|_{C^1} \leq \min(m_1 M, m_2 M)\},$$

where

$$m_1 := \max_{t \in [0,1]} \int_0^1 |G(t, s)| ds,$$

$$m_2 := \max_{t \in [0,1]} \int_0^1 \left| \frac{\partial G}{\partial t}(t, s) \right| ds,$$

and M is defined as in (4). It is clear that D is a closed, bounded, and convex set in $C^1[0, 1]$ and one can show that $T : D \to D$ is a completely continuous mapping by Arzelà-Ascoli theorem. By applying Schauder's fixed point theorem, we obtain that T has a fixed point u in D, which is a solution of problem (5), (2). From Proposition 1, this fixed point u of T is indeed a solution of equation

$$u''(t) + \tilde{f}(t, u(t), u'(t)) = 0,$$

with boundary condition (2). Next, we claim that $|u'(t)| \leq N$ on $[0, 1]$. It follows from Mean Value Theorem, and $\alpha(t) \leq u(t) \leq \beta(t)$ on $[0, 1]$ that there exists $x_0 \in (0, 1)$ such that

$$|u'(x_0)| = \max\{|\alpha(0) - \beta(1)|, |\alpha(1) - \beta(0)|\} := \lambda < N.$$

If the claim does not hold, then there exists $[x_3, x_1] \subseteq [0, 1]$, such that one of the following case hold:

(i) $u'(x_3) = -N$, $u'(x_1) = -\lambda$, and $-N < u'(t) < -\lambda$, for $t \in (x_3, x_1)$,

(ii) $u'(x_3) = -\lambda$, $u'(x_1) = -N$, and $-N < u'(t) < -\lambda$, for $t \in (x_3, x_1)$,

(iii) $u'(x_3) = N$, $u'(x_1) = \lambda$, and $\lambda < u'(t) < N$, for $t \in (x_3, x_1)$,

(iv) $u'(x_3) = \lambda$, $u'(x_1) = N$, and $\lambda < u'(t) < N$, for $t \in (x_3, x_1)$.

Let us consider Case (i), other cases are discussed similarly. On $[x_3, x_1]$, we infer that

$$|u''(t)| = |\widetilde{f}(t, u(t), u'(t))| = |f(t, u(t), u'(t))| \leq h(|u'(t)|)$$

on $[x_0, x_1]$. Thus,

$$u(x_3) - u(x_1) = \int_{x_3}^{x_1} - u'(t)\,dt \geq \int_{x_3}^{x_1} \frac{-u'(t)|u''(t)|}{h(-u'(t))}\,dt$$

$$\geq \int_{\lambda}^{N} \frac{s}{h(s)}\,ds > \max_{t\in[0,1]} \beta(t) - \min_{t\in[0,1]} \alpha(t),$$

a contradiction. Hence, this fixed point of T is indeed a solution of problem (1), (2), and the proof is completed. □

4. Carathéodory Case

In this section, we first introduce the definition of Carathéodory function, and the notion of $W^{2,1}$-upper (lower) solution. Then, we shall discuss the existence of $W^{2,1}$-solution by assuming the existence of upper and lower solutions under Carathéodory case.

Definition 4. A function $f(t, u, v)$ defined on $E \subseteq [a, b] \times \mathbb{R} \times \mathbb{R}$ is called a *Carathéodory function* on *E*, if

(i) for almost every $t \in [a, b]$, $f(t, \cdot, \cdot)$ is continuous on its domain,

(ii) for any $u, v \in \mathbb{R}$, the function $f(\cdot, u, v)$ is measurable on its domain,

(iii) for any $r > 0$, there exists $p_r \in L^1(a, b)$ such that for any $u, v \in [-r, r]$, and for almost every $t \in [a, b]$ with $(t, u, v) \in E$, we have $|f(t, u, v)| \leq p_r(t)$.

Now, we impose $0 < \delta \leq 1$ on the boundary condition (2) and start concerning about Carathéodory case, that is, the Carathéodory source term *f*. We shall discuss the existence of $W^{2,1}$-solution by assuming the existence of upper and lower solutions.

4.1. Existence of $W^{2,1}$-solutions. In this subsection, we impose an additional assumption on the increasing property of *f* and discuss the existence of $W^{2,1}$-solution. We first introduce the definitions of $W^{2,1}$-upper and lower solutions as below.

Definition 5. A function $\alpha \in C[0, 1]$ is called a $W^{2,1}$-*lower solution* of problem (1), (2), if it satisfies

(i) $\alpha(0) \leq 0$, $\alpha(1) \leq \delta\alpha(\eta)$, and

(ii) for any $t_0 \in (0, 1)$, either $D^{-}\alpha(t_0) < D_{+}\alpha(t_0)$, or there exists an open interval $I_0 \subseteq (0, 1)$ containing t_0 such that $\alpha \in W^{2,1}(I_0)$, and for almost every $t \in I_0$, we have

$$\alpha''(t) + f(t, \alpha(t), \alpha'(t)) \geq 0.$$

Definition 6. A function $\beta \in C[0, 1]$ is called a $W^{2,1}$-*upper solution* of problem (1), (2), if it satisfies

(i) $\beta(0) \geq 0$, $\beta(1) \geq \delta\beta(\eta)$, and

(ii) for any $t_0 \in (0, 1)$, either $D_{-}\beta(t_0) > D^{+}\beta(t_0)$, or there exists an open interval $I_0 \subseteq (0, 1)$ containing t_0 such that $\beta \in W^{2,1}(I_0)$, and for almost every $t \in I_0$, we have

$$\beta''(t) + f(t, \beta(t), \beta'(t)) \leq 0.$$

For this pair of $W^{2,1}$-upper and lower solutions, we can define the Nagumo's condition for Carathéodory function similar as Definition 3, and conclude.

Lemma 3. *Assume that Carathéodory f is a Carathéodory function satisfying Nagumo's condition on* $[0, 1]$ *with respect to* α *and* β. *Then for any solution* $u \in W^{2,1}(0, 1)$ *of* (1) *with* $\alpha(t) \leq u(t) \leq \beta(t)$ *on* $[0, 1]$, *there exists an* $L > 0$ *depending only on* α, β, h *such that*

$$|u'(t)| \leq L \text{ on } [0, 1].$$

Proof. All arguments are as same as Bernfeld and Lakshmikantham's proof, see [2], except one point that we use the more general version of change of variables in integral, which is mentioned in [1, p.165]. □

Proposition 2. *Let* $\alpha(t)$ *and* $\beta(t)$ *be the respective* $W^{2,1}$*-lower and upper solution of problem* (1), (2) *with* $\alpha(t) \leq \beta(t)$ *on* $[0, 1]$, *and let f be a Carathéodory function on E, and satisfy Nagumo's condition with respect to* α *and* β *on* $[0, 1]$, *where*

$$E := \{(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \mid \alpha(t) \leq u \leq \beta(t)\}.$$

Moreover, we assume that on domain E, for each fixed (t, u), $f(t, u, v)$ *is nondecreasing in v. If* $u \in W^{2,1}(0, 1)$ *is a solution of the modified problem* (5), (2), *then* $\alpha(t) \leq u(t) \leq \beta(t)$, *for any* $t \in [0, 1]$.

Proof. Suppose there exists $t_0 \in [0, 1]$ such that

$$\min_{t \in [0,1]} (u(t) - \alpha(t)) = u(t_0) - \alpha(t_0) < 0.$$

Case (i). If $t_0 \in (0, 1)$, we have $u'(t_0) - D^{-}\alpha(t_0) \leq u'(t_0) - D_{+}\alpha(t_0)$, which implies $D^{-}\alpha(t_0) \geq D_{+}\alpha(t_0)$. Hence, by Definition 5 and the continuity of $u - \alpha$ at t_0, there exist an open interval $I_0 \subseteq (0, 1)$ with $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$, and a neighborhood G of t_0 contained in I_0, such that for almost every $t \in G$,

$$u(t) - \alpha(t) < 0,$$

and

$$\alpha''(t) + f(t, \alpha(t), \alpha'(t)) \geq 0.$$

Furthermore, it follows from $u'(t) - \alpha'(t) \geq 0$, for $t \geq t_0$, $t \in G$ that

$$\tilde{\tilde{f}}(t, u(t), u'(t)) = \begin{cases} f(t, \alpha(t), u'(t)) + \frac{\alpha(t) - u(t)}{1 + |\alpha(t)| + |u(t)|}, & \text{if } |u'(t)| \leq N, \\ f(t, \alpha(t), N) + \frac{\alpha(t) - u(t)}{1 + |\alpha(t)| + |u(t)|}, & \text{if } u'(t) > N. \end{cases}$$

Since, the nondecreasing assumption on f, one can conclude that for $t \geq t_0$,

$$\begin{aligned} u'(t) - \alpha'(t) &= \int_{t_0}^{t} (u''(s) - \alpha''(s))\, ds \\ &\leq \int_{t_0}^{t} -\tilde{\tilde{f}}(s, u(s), u'(s)) + f(s, \alpha(s), \alpha'(s))\, ds \\ &< 0. \end{aligned}$$

This implies that the minimum of $u - \alpha$ can not occur at t_0, a contradiction.

Case (ii). If $t_0 = 0$, by the definition of $W^{2,1}$-lower solution $\alpha(0) \leq 0$, we then have

$$0 = u(0) \leq u(0) - \alpha(0) < 0,$$

and get a contradiction.

Case (iii). If $t_0 = 1$, it follows from the conclusion of Case (i), and $0 < \delta \leq 1$ that

$$u(1) - \alpha(1) \geq \delta(u(\eta) - \alpha(\eta)) > \delta(u(1) - \alpha(1)) \geq u(1) - \alpha(1),$$

which is impossible.

Consequently, we obtain $\alpha(t) \leq u(t)$ on $[0, 1]$. By the similar arguments as above, we also have

$$u(t) \leq \beta(t) \text{ on } [0, 1].$$

$\square$

Theorem 2. *Let* $\alpha(t)$ *and* $\beta(t)$ *be the respective* $W^{2,1}$*-lower and upper solution of problem* (1), (2) *with* $\alpha(t) \leq \beta(t)$ *on* $[0, 1]$, *and let f be a Carathéodory function on E and satisfy Nagumo's condition with respect to* α *and* β *on* $[0, 1]$, *where*

$$E := \{(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \mid \alpha(t) \leq u \leq \beta(t)\}.$$

Moreover, we assume that on domain E, for each fixed (t, u), $f(t, u, v)$ *is nondecreasing in v. Then, the problem* (1), (2), *has at least one solution* $u \in W^{2,1}$ *such that, for all* $t \in [0, 1]$,

$$\alpha(t) \leq u(t) \leq \beta(t).$$

Proof. As in the proof of Theorem 1, we consider the modified problem (5), (2) with respect to the given $\alpha(t)$ and $\beta(t)$. Consider the Banach space $C^1[0, 1]$ with usual C^1-norm, and the operator $T : C^1[0, 1] \to C^1[0, 1]$ by (6). Since f is a Carathéodory function, for $r >> 0$, there exists a function $h_r \in L^1(0, 1)$ such that for any $u \in [-r, r]$, for almost every $t \in [0, 1]$ with $(t, u, v) \in E$, we have

$$|f(t, u, v)| \leq h_r(t).$$

Define

$$K := \{u \in C^1[0, 1] \mid \|u\| \leq \min(M_1, M_2)\},$$

where

$$M_1 := \max_{t\in[0,1]} \int_0^1 |G(t, s)| [h_r(s) + 1] ds < \infty,$$

and

$$M_2 := \max_{t\in[0,1]} \int_0^1 \left| \frac{\partial G}{\partial t}(t, s) \right| [h_r(s) + 1] ds < \infty.$$

It is clear that K is a closed, bounded, and convex set in $C^1[0, 1]$ and one can show that $T : K \to K$ is a completely continuous mapping by Arzelà-Ascoli theorem, and Lebesgue dominated convergence theorem. By applying Schauder's fixed point theorem, we obtain that T has a fixed point in K, which is a solution of problem (5), (2). From Proposition 2 and similar arguments in Theorem 1, this fixed point of T is also a solution of problem (1), (2). Hence, we complete the proof. □

4.2. Non-tangency solution. In this subsection, we afford another stronger $W^{2,1}$-lower and upper solutions to get a strict inequality of the solution between them.

Definition 7. A function $\alpha \in C[0, 1]$ is a strict $W^{2,1}$-lower solution of problem (1), (2), if it is not a solution of problem (1), (2), $\alpha(0) < 0$, $\alpha(1) \le \delta\alpha(\eta)$, and for any $t_0 \in (0, 1)$, one of the following is satisfied:

(i) $D^-\alpha(t_0) < D_+\alpha(t_0)$,

(ii) there exist an interval $I_0 \subseteq [0, 1]$ and $\epsilon > 0$ such that $t_0 \in \operatorname{int}(I_0)$, $\alpha \in W^{2,1}(I_0)$, and for almost every $t \in I_0$, for all $u \in [\alpha(t), \alpha(t) + \epsilon]$, $u'(t) \in \mathbb{R}$, we have

$$\alpha''(t) + f(t, u(t), u'(t)) \ge 0.$$

Definition 8. A function $\beta \in C[0, 1]$ is a strict $W^{2,1}$-upper solution of problem (1), (2), if it is not a solution of problem (1), (2), $\beta(0) > 0$, $\beta(1) \ge \delta\beta(\eta)$, and for any $t_0 \in (0, 1)$, one of the following is satisfied:

(i) $D_-\beta(t_0) > D^+\beta(t_0)$,

(ii) there exist an interval $I_0 \subseteq [0, 1]$ and $\epsilon > 0$ such that $t_0 \in \text{int}(I_0)$, $\beta \in W^{2,1}(I_0)$, and for almost every $t \in I_0$, for all $u \in [\beta(t) - \epsilon, \beta(t)]$, $u'(t) \in \mathbb{R}$, we have

$$\beta''(t) + f(t, u(t), u'(t)) \leq 0.$$

Remark. Every strict $W^{2,1}$-lower (upper) solution of problem (1), (2) is a $W^{2,1}$-lower (upper) solution.

We also consider the three-point boundary value problem (1), (2), which can be written in the form

$$u(t) = (Mu)(t) := \int_0^1 G(t, s) f(s, u(s), u'(t)) ds, \tag{7}$$

where $G(t, s)$ is defined by (3).

Now, we are going to show that the solution curve of problem (1), (2) can not be tangent to upper or lower solutions from below or above.

Proposition 3. *Let* $\alpha(t)$ *and* $\beta(t)$ *be the respective strict* $W^{2,1}$*-lower and upper solution of problem* (1), (2) *with* $\alpha(t) \leq \beta(t)$ *on* $[0, 1]$, *and let* f *be a Carathéodory function on* E *and satisfy Nagumo's condition with respect to* α *and* β *on* $[0, 1]$, *where*

$$E := \{(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \mid \alpha(t) \leq u \leq \beta(t)\}.$$

If $u \in W^{2,1}(0, 1)$ *is a solution of problem* (1), (2) *with* $\alpha \leq u \leq \beta$ *on* $[0, 1]$, *then* $\alpha(t) < u(t) < \beta(t)$, *for any* $t \in [0, 1]$.

Proof. As α is not a solution, u is not identical to α. Assume the conclusion does not hold, then

$$t_0 := \inf\{t \in [0, 1] \mid u(t) = \alpha(t)\}$$

exists. Since $u - \alpha$ has minimum at t_0, we have $D^{-}\alpha(t_0) \geq D_{+}\alpha(t_0)$ and $u(t_0) - \alpha(t_0) = 0$.

Case (i). If $t_0 \in (0, 1)$, according to the Definition 7, there exist I_0, $\epsilon_0 > 0$, and $t_1 \in I_0$ with $t_1 < t_0$ such that, for every $t \in (t_1, t_0)$, $u(t) \leq \alpha(t) + \epsilon_0$, $u'(t_1) - \alpha'(t_1) < 0$, and for a.e. $t \in (t_1, t_0)$

$$\alpha''(t) + f(t, u(t), u'(t)) \geq 0.$$

Hence, we have the contradiction since

$$0 < (u' - \alpha')(t_0) - (u' - \alpha')(t_1) = -\int_{t_1}^{t_0} [f(t, u(t), u'(t)) + \alpha''(t)]dt \leq 0.$$

Case (ii). If $t_0 = 0$, by the definition of strict $W^{2,1}$-lower solution that $\alpha(0) < 0$, we then have

$$0 = u(0) - \alpha(0) < 0,$$

and get a contradiction.

Case (iii). If $t_0 = 1$, repeat the same arguments in Case (iii) of Proposition 2.

Therefore, we obtain $\alpha(t) < u(t)$ on $[0, 1]$. The inequality $u(t) < \beta(t)$ on $[0, 1]$ can be proved by the similar arguments as above. □

Theorem 3. *Let* $\alpha(t)$ *and* $\beta(t)$ *be the respective strict* $W^{2,1}$*-lower and upper solution of problem* (1), (2) *with* $\alpha(t) \leq \beta(t)$ *on* $[0, 1]$, *and let f be a Carathéodory function on E and satisfy Nagumo's condition with respect to* α *and* β *on* $[0, 1]$, *where*

$$E := \{(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \mid \alpha(t) \leq u \leq \beta(t)\}.$$

If, in addition, we assume that on domain E, for each fixed (t, u), $f(t, u, v)$ *is nondecreasing in v. Then,*

$$\deg(I - M, \Omega) = 1,$$

where M is defined by (7), *and*

$$\Omega := \{u \in C^1[0, 1] \mid \alpha(t) < u(t) < \beta(t), -N - 1 < u'(t) < N + 1, \textit{for any } t \in [0, 1]\}.$$

As a result, problem (1), (2), *has at least one solution* $u \in W^{2,1}(0, 1)$ *such that, for any* $t \in [0, 1]$

$$\alpha(t) < u(t) < \beta(t).$$

Proof. Consider the modified problem (5), (2), and the operator $T : C^1[0, 1] \to C^1[0, 1]$ is defined by

$$(Tu)(t) = \int_0^1 G(t, s)\tilde{\tilde{f}}(s, u(s), u'(s))ds,$$

where $G(t, s)$ is defined by (3). It is clear that ImT is bounded, hence, for any $R > 0$ large enough, ImT is contained in the ball $B_R(0)$ with center at the origin and radius R.

For any $\lambda \in [0, 1]$, $x \in B_R(0)$, let

$$H(\lambda, x) = \lambda(x - Tx) + (1 - \lambda)x.$$

One can demonstrate that $H(\lambda, x)$ is a homotopy and $H(\lambda, x) \neq 0$, for all $x \in \partial B_R(0)$ and $0 \leq \lambda \leq 1$. From the homotopy invariance of degree theory, we have

$$\deg(I - T, B_R(0)) = \deg(I, B_R(0)) = 1.$$

It follows from Remark and Proposition 2 that every solution u of problem (5), (2) satisfies $\alpha(t) \leq u(t) \leq \beta(t)$ on $[0, 1]$, moreover, by Proposition 3 and similar demonstration as in the proof of Theorem 1, we conclude that $\alpha(t) < u(t) < \beta(t)$ on $[0, 1]$ and $|u'(t)| \leq N$. This proves that such a solution is in Ω.

As M and T coincide on $\overline{\Omega}$, we obtain

$$\deg(I - M, \Omega) = \deg(I - T, \Omega).$$

By the excision property of degree theory, we get

$$\deg(I - T, \Omega) = \deg(I - T, B_R(0)) = 1.$$

Hence, we complete the proof. □

5. Examples

In this section, we afford two examples as applications of our results in Theorems 1 and 2.

Example 1. Consider the second order three-point BVP

$$u''(t) + \exp(-u(t)) - u(t)|u'(t)| + t - 1 = 0, \; t \in (0, 1), \tag{8}$$

$$u(0) = 0, \; u(1) = cu(1/2), \tag{9}$$

where $0 < c \leq 1$. We observe that $\alpha(t) = 0$ and $\beta(t) = t$ are lower and upper solutions of problem (8), (9), respectively. Clearly, $f(t, u, v) := \exp(-u) - u|v| + t - 1$ is continuous on $[0, 1] \times \mathbb{R} \times \mathbb{R}$. Also, f satisfies the Nagumo's condition with respect to α and β on $[0, 1]$, where $h(s) := 2 + s$ on $[0, \infty)$. All assumptions in Theorem 1 are satisfied, and hence, we obtain a classical solution u such that $0 \leq u(t) \leq t$ on $[0, 1]$.

Example 2. Consider the second order differential equation

$$u''(t) - \sin u(t) - g(t) + |u'(t)|^{\alpha-1} u'(t) \cos u(t) = 0, \; t \in (0, 1), \tag{10}$$

equipped with (9), where $0 < c \leq 1, 1 \leq \alpha \leq 2,$ and

$$g(t) = \begin{cases} 1, & \text{if } t \in [0, 1] \cap \mathbb{Q}, \\ \sin t, & \text{if } t \in [0, 1] \cap \mathbb{Q}^c. \end{cases}$$

We observe that $\alpha(t) = -t$ and $\beta(t) = 0$ are $W^{2,1}$-lower and upper solutions of problem (10), (9), respectively. Clearly, $f(t, u, v) := -\sin u + g(t) + |v|^{\alpha-1} v \cos u$ is a Carathéodory function on E, where

$$E := \{(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \mid -t \leq u \leq 0\}.$$

Also, f satisfies the Nagumo's condition with respect to α and β on $[0, 1]$, and f is increasing with respective to v for $t \in [0, 1]$ and $u \in [-1, 0]$. All assumptions in Theorem 2 are satisfied. Hence, we get a $W^{2,1}$-solution u such that $-t \leq u(t) \leq 0$ on $[0, 1]$.

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